SELF-AFFINE SPECTRAL MEASURES AND FRAME SPECTRAL MEASURES ON $\mathbb{R}^d$

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ABSTRACT. We study Fourier bases on invariant measures generated by affine iterated function systems in $\mathbb{R}^d$ with integer coefficients. We show that, for simple digit sets, these systems satisfy the open set condition and have no overlap. We present natural geometric conditions under which such measures have an orthonormal basis or a frame of exponential functions with frequencies being a subset of $\mathbb{Z}^d$. Moreover, we characterize when such measures have a spectrum in $\mathbb{Z}^d$.

CONTENTS

1. INTRODUCTION

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product. We say that μ is a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ called *spectrum* such that $E(\Lambda) := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. The Fourier transform of μ is defined to be

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, x \rangle} d\mu(x).$$

It is direct to verify that a measure is a spectral measure with spectrum Λ if and only if the following two conditions are satisfied:

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(i) (Orthogonality) $\widehat{\mu}(\lambda - \lambda') = 0$ for all distinct $\lambda, \lambda' \in \Lambda$ and

(ii) (Completeness) If for $f \in L^2(\mu)$, $\int f(x) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) = 0$ for all $\lambda \in \Lambda$, then $f = 0$.

The study of spectral measures was initiated by Fuglede in 1974 [Fu], by investigating which subsets of $\mathbb{R}^d$ with the Lebesgue measures are spectral. The research was advanced into the realm of fractals when Jorgensen and Pedersen discovered the first singular spectral measure [JP]. They showed that the one-fourth Cantor measure is a spectral measure while the one-third Cantor measure is not. Following these discoveries, the theory of spectral measures was further developed into many different facets [St1, St2, ŁaW, Li, DJ2, DHJ, DHS, Da, DaHL, HuL]. All these work attempted to give a satisfactory answer to the following question:

Question 1: When is a measure μ spectral? When is a countable set a spectrum of μ?

This problem lies at the interface between analysis, geometry and number theory and it relates to translational tilings. Fuglede conjectured that a set in $\mathbb{R}^d$ is spectral if and only if it is a translational tile, but the conjecture was disproved by Tao and others on dimension 3 or higher [T, KM]. On $\mathbb{R}^1$, Fuglede's conjecture can be solved if one settles some number theoretic conditions [CM, Ła, DL2]. Recently, Gabardo and Lai [GL] discovered that any measures μ, for which one can find another measure ν, such that the convolution $\mu * \nu$ is the Lebesgue measure on $[0,1]^d$, are spectral measures. This yields large numbers of examples of spectral measures essentially arising from "translational tilings" in a generalized sense, indicating also that general spectral measures are associated in a certain way with translational tilings.

There is an algebraic condition linking translational tilings and spectral measures. This condition has been known to be a key algebraic criterion to produce spectral measures and it can be obtained in many cases from translational tilings of the integers.

Definition 1.1. Let $R \in M_d(\mathbb{Z})$ be an $d \times d$ expansive matrix (all eigenvalues have modulus strictly greater than 1) with integer entries. Let $B, L \subset \mathbb{Z}^d$ be a finite set of integer vectors with $N = \#B = \#L$ (# means the cardinality). We say that the system (R, B, L) forms a *Hadamard triple* (or $(R^{-1}B, L)$ forms a *compatible pair* in [ŁaW]) if the matrix

$$(1.1) \qquad\qquad H = \frac{1}{\sqrt{N}} \left[e^{2\pi i \langle R^{-1}b, \ell \rangle} \right]_{\ell \in L, b \in B}$$

is unitary, i.e., $H^*H = I$.

The system (R, B, L) forms a Hadamard triple if and only if the Dirac measure $\delta_{R^{-1}B} = \frac{1}{\#B} \sum_{b \in B} \delta_{R^{-1}b}$ is a spectral measure with spectrum L. Moreover, this property is an important property to produce examples of singular spectral measures, particularly the self-affine measures.

Definition 1.2. For a given expansive $d \times d$ integer matrix R and a finite set of integer vectors B with $\#B =: N$, we define the *affine iterated function system* (IFS) $\tau_b(x) = R^{-1}(x + b)$, $x \in \mathbb{R}^d, b \in B$. The *self-affine measure* (with equal weights) is the unique probability measure $\mu = \mu(R, B)$ satisfying

$$(1.2) \qquad \mu(E) = \sum_{b \in B} \frac{1}{N} \mu(\tau_b^{-1}(E)), \text{ for all Borel subsets } E \text{ of } \mathbb{R}^d.$$

This measure is supported on the *attractor* $T(R, B)$ which is the unique compact set that satisfies

$$T(R, B) = \bigcup_{b \in B} \tau_b(T(R, B)).$$

The set $T(R, B)$ is also called the *self-affine set* associated with the IFS. See [Hut81] for details.

The following conjecture about spectral self-affine measure is not yet settled.

Conjecture 1.3. *Let $\mu = \mu(R, B)$ be a self-affine measure. Suppose that we can find $L \subset \mathbb{Z}^d$ such that (R, B, L) forms a Hadamard triple. Then μ is a spectral measure.*

It is fairly easy to construct an infinite mutually orthogonal set of exponential functions using the Hadamard triple assumption. However, checking these exponentials form a complete set in $L^2(\mu)$ is a much more difficult task. When $d = 1$, Conjecture 1.3 was solved by Łaba and Wang [ŁaW] and refined in [DJ1]. The situation becomes more complicated when $d > 1$. Dutkay and Jorgensen showed that the conjecture is true if (R, B, L) satisfies a technical condition called *reducibility condition*, but this condition requires a very symmetric structure on B and L [DJ2].

In this paper, we would like to provide a natural geometric criterion guaranteeing that Conjecture 1.1 holds. Moreover, we characterize when the self-affine measures have a spectrum in $\mathbb{Z}^d$. We observe that if (R, B, L) forms a Hadamard triple, then the elements of B must be distinct as residue classes (mod $R(\mathbb{Z}^d)$). Let $\overline{B}$ be a complete set of representatives (mod $R(\mathbb{Z}^d)$) containing B. Then the attractor $T = T(R, \overline{B})$ is a translational tile. This tile is called a *self-affine tile*. We refer readers to [LW1] and the survey [W] for the theory of self-affine tiles. One of the important results we need is that this tile admits a lattice tiling of $\mathbb{R}^d$ with some lattice $\Gamma \subset \mathbb{Z}^d$ [LW2]. We say that T is a Γ-*tile* if T tiles $\mathbb{R}^d$ by the lattice Γ. Denote by T° the interior of T.

Definition 1.4. Let R be a $d \times d$ integer matrix. We call a finite set $B \subset \mathbb{Z}^d$, a *simple digit set for R*, if distinct elements of B are not congruent (mod $R(\mathbb{Z}^d)$).

We say that the iterated function system $\{\tau_b\}_{b\in B}$ satisfies the *open set condition* (OSC) if there exists a non-empty open set U such that

$$\tau_b(U) \cap \tau_{b'}(U) = \emptyset, \text{ and } \bigcup_{b\in B} \tau_b(U) \subset U.$$

We say that the iterated function system $\{\tau_b\}_{b\in B}$ satisfies the *strong open set condition* (SOSC) if we can furthermore choose the open set U such that $U \cap T(R, B) \neq \emptyset$.

We say that the measure $\mu = \mu(R, B)$ *has no overlap* if

$$\mu(\tau_b(T(R, B)) \cap \tau_{b'}(T(R, B))) = 0 \text{ for all } b \neq b' \text{ in } B.$$

The following theorem was proved by He and Lau [HL, Theorem 4.4], see also [Sc] for self-similar IFSs.

Theorem 1.5. [HL] *For a self-affine IFS, OSC and SOSC are equivalent.*

We introduce the following separation condition, which we will see, is closely related to spectral measures:

Definition 1.6. We say that the IFS $\{\tau_b\}_{b\in B}$ satisfies the *T-strong open set condition* (denoted in short by *T-SOSC*) if there exists a complete set of representatives $(\text{mod } R(\mathbb{Z}^d))$, $\overline{B}$ such that $B \subset \overline{B}$, $T(R, B) \cap T^\circ \neq \emptyset$ where $T = T(R, \overline{B})$.

As shown in [LW2], if $\overline{B}$ is a complete set of representatives $(\text{mod } R(\mathbb{Z}^d))$ then $T(R, \overline{B})$ tiles $\mathbb{R}^d$ by some lattice so it is a *self-affine* tile.

First we present a result about these separation conditions in this context.

Theorem 1.7. *Let R be a $d \times d$ expansive integer matrix and let B be a simple digit set for R. Then the affine iterated functions system associated to R and B satisfies the OSC, SOSC and the no overlap condition.*

Let $\mu = \mu(R, B)$ be the associated self-affine measure. Consider the following conditions:

 (i) *The affine IFS associated with R and B satisfies the (T-SOSC) with T a $\mathbb{Z}^d$-tile.*
 (ii) *For all $k \neq k'$ in $\mathbb{Z}^d$, $\mu((T(R, B) + k) \cap (T(R, B) + k')) = 0$.*
 (iii) *The set*

$$(1.3) \qquad\qquad \mathcal{Z} := \{\xi \in \mathbb{R}^d : \widehat{\mu}(\xi + k) = 0, \text{ for all } k \in \mathbb{Z}^d\}$$

   is empty.

Then we have the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

We denote by

$$(1.4) \qquad B_n := B + RB + R^2 B + \dots + R^{n-1} B = \left\{ \sum_{j=0}^{n-1} R^j b_j : b_j \in B \right\}.$$

Let $\mathbb{Z}[R, B]$ be the smallest R-invariant lattice containing all B_n (invariant means $R(\mathbb{Z}[R, B]) \subset \mathbb{Z}[R, B]$). By Proposition 4.1, for the study of Conjecture 1.3, there is no loss of generality if we assume that $\mathbb{Z}[R, B] = \mathbb{Z}^d$.

The next result gives a partial resolution to Conjecture 1.3.

Theorem 1.8. *Suppose that (R, B, L) forms a Hadamard triple and $\mathbb{Z}[R, B] = \mathbb{Z}^d$ and let $\mu = \mu(R, B)$ be the associated self-affine measure $\mu = \mu(R, B)$. Then the following are equivalent*

(i) *$\mathcal{Z} = \emptyset$, where $\mathcal{Z}$ is defined in (1.3).*
(ii) *μ has a spectrum in $\mathbb{Z}^d$.*

In particular, if $\mathcal{Z} = \emptyset$, then μ is a spectral measure.

The above theorems lead to the following corollary.

Corollary 1.9. *Let $R = diag(N_1, ..., N_d)$, where $N_i \geq 2$ are integers and B is a set of integer vectors contained in $\prod_{i=1}^{d} \{0, 1, ..., N_i - 1\}$. Suppose that (R, B, L) forms a Hadamard triple with some L. Then the self-affine measure μ in (1.2) is a spectral measure with a spectrum in $\mathbb{Z}^d$.*

The theorems and the corollary generalizes many previous work. First, it gives us a sufficient condition on the dimensions for which the generalized Sierpinski gasket is spectral, see Example 5.2, while most of the attentions previously focus only on dimension $d = 2, 3$. Furthermore, We recover the results of Łaba-Wang [ŁaW], and Dutkay-Jorgensen [DJ1], showing that Conjecture 1.1 holds on $\mathbb{R}^1$, see Example 5.1. Moreover, it connects to the research on the topology of self-affine tiles, see Example 5.3.

The proof of Theorem 1.8 is different from the other proofs of completeness in the literature [ŁaW, St1, DJ2, DaHL, Da], in which authors established the completeness property by checking the Jorgensen-Pedersen criterion (i.e. $\sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2 = 1$). We resort to an approach from matrix analysis by relaxing the Hadamard triple condition to the following condition.

Definition 1.10. We say that the pair (R, B) satisfies the *almost-Parseval-frame condition* if for any $\epsilon > 0$, there exists n and a subset $J_n \subset \mathbb{Z}^d$ such that

$$(1.5) \qquad (1 - \epsilon) \sum_{b \in B_n} |w_b|^2 \leq \sum_{\lambda \in J_n} \left| \frac{1}{\sqrt{N^n}} \sum_{b \in B_n} w_b e^{-2\pi i \langle R^{-n} b, \lambda \rangle} \right|^2 \leq (1 + \epsilon) \sum_{b \in B_n} |w_b|^2$$

for all $\mathbf{w} = (w_b)_{b \in B_n} \in \mathbb{C}^{N^n}$. Equivalently,

$$(1 - \epsilon) \|\mathbf{w}\|^2 \leq \|F_n \mathbf{w}\|^2 \leq (1 + \epsilon) \|\mathbf{w}\|^2$$

where $F_n = \left[\frac{1}{\sqrt{N^n}}e^{-2\pi i \langle (R^{-n}b),\lambda\rangle}\right]_{\lambda \in J_n, b \in B_n}$ and $\|\cdot\|$ denotes the Euclidean norm.

Hadamard triples do satisfy this condition and we can prove another general theorem.

Theorem 1.11. *Suppose that B is simple digit set for R and that (R, B) satisfies the almost-Parseval-frame condition. Assume in addition that $\mathbb{Z}[R, B] = \mathbb{Z}^d$. Then the set $\mathcal{Z}$ defined in (1.3) is empty if and only if the self-affine measure $\mu = \mu(R, B)$ admits a Fourier frame $E(\Lambda) = \{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ with $\Lambda \subset \mathbb{Z}^d$, i.e., there exists $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{\lambda \in \Lambda} |\int f(x)e^{-2\pi i \langle \lambda, x\rangle}d\mu(x)|^2 \le B\|f\|^2, \ \forall \ f \in L^2(\mu).$$

It is clear that the Fourier frames are a natural generalization of exponential orthonormal bases. Whenever Fourier frames exist, μ is called a *frame spectral measure* and Λ is called a *frame spectrum.* Some of the fundamental properties of Fourier frames were investigated in [HLL, DHW, DL1]. Theorem 1.11 gives a new sufficient condition for an answer to the following question.

Question 2: Does a self-affine measure still admit a Fourier frame even though it is not a spectral measure?

For the simplest case, can we construct a Fourier frame on the one-third Cantor measure? While the one-third Cantor set satisfies clearly the (T-SOSC) is satisfied by choosing $T = [0, 1]$, using Theorem 1.7 and 1.11, this problem is changed to a matrix analysis problem, which is to construct finite sets J_n so that the almost-Parseval-frame condition holds. At this time, we are unable to give a full solution. However, the recent solution of the Kadison-Singer conjecture [MSS] enabled Nitzan, Olevskii, Unlanovskii [NOU] to construct Fourier frames on unbounded sets of finite measures. One of their lemmas gives us a weak solution:

Proposition 1.12. *For the same definition of (R, B) in Definition 1.10. There exist universal constants $0 < c_0 < C_0 < \infty$ such that for all n, there exists J_n such that*

$$c_0 \sum_{b \in B_n} |w_b|^2 \le \sum_{\lambda \in J_n} \left|\frac{1}{\sqrt{N^n}}\sum_{b \in B_n} w_b e^{-2\pi i \langle R^{-n}b, \lambda\rangle}\right|^2 \le C_0 \sum_{b \in B_n} |w_b|^2$$

for all $(w_b)_{b \in B_n} \in \mathbb{C}^{N^n}$.

Unlike the proof in Theorem 1.11, we are unable to concatenate the J_n in Proposition 1.12. It would be nice if we can make J_n an increasing subsets.

We organize the paper as follows. In Section 2, we discuss the basic theory of self-affine measure and prove Theorem 1.7. In Section 3, we discuss the almost Parseval frame condition and the Hadamard triple condition. In Section 4, we prove Theorems 1.8 and 1.11. In Section

5, we present the examples based on Theorem 1.8. In Section 6, we discuss some general follow-up problems related to the questions and conjectures we posed.

2. Self-affine measures

Let R be an expansive matrix with integer entries and let B be a simple digit set for R. Define

$$\tau_b(x) = R^{-1}(x + b), \ b \in B.$$

and let $T = T(R, B)$ be its attractor.

Let us introduce some multi-index notation. Let $B^n = B \times B... \times B$ (n copies) and $\Sigma = \bigcup_{n=1}^{\infty} B^n$. For each $\mathbf{b} = (b_1, ..., b_n) \in B^n$,

$$\tau_{\mathbf{b}}(x) = \tau_{b_1} \circ ... \circ \tau_{b_n}(x).$$

Also for any set $A \subset \mathbb{R}^d$, we define $A_{\mathbf{b}} = \tau_{\mathbf{b}}(A)$. Given a set of probabilities $0 < p_b < 1$, $b \in B$, ($\sum_{b \in B} p_b = 1$), the associated *self-affine measure* is the unique Borel probability measure supported on $T(R, B)$ satisfying the invariance identity

$$(2.1) \qquad\qquad \mu = \sum_{b \in B} p_b \mu_b,$$

where we define $\mu_b(E) = \mu(\tau_b^{-1}(E))$, for all Borel sets E, see [Hut81]. By iterating the identity, we have

$$\mu = \sum_{\mathbf{b} \in B^n} p_{\mathbf{b}} \mu_{\mathbf{b}},$$

where $p_{\mathbf{b}} = p_{b_1}....p_{b_n}$ and $\mu_{\mathbf{b}}(E) = \mu(\tau_{\mathbf{b}}^{-1}(E))$ for all Borel sets E and $\mathbf{b} = (b_1, ..., b_n)$.

For any set F, we denote by $\overline{F}$, T^o ∂F the closure, interior and its boundary. In the study of spectral measures, the no overlap condition for a self-affine measures is very important. The following theorem shows that the strong open set condition implies the no overlap condition. Its proof is motivated by [DeHL, Lemma 2.2]

Theorem 2.1. *Suppose that the IFS satisfies the strong open set condition with the open set U. Then for any self-affine measure in (2.1,) $\mu(U) = 1$ and $\mu(\partial U) = 0$. Moreover, μ satisfies the no overlap condition.*

Proof. As $T(R, B) \cap U \neq \emptyset$, we can find $x_0 \in T(R, B) \cap U$ and $\delta > 0$ such that $B_\delta(x_0) \subset U$. In particular, there exists $\mathbf{b_0} \in B^n$, for some n such that $\tau_{\mathbf{b_0}}(T(R, B)) \subset B_\delta(x_0) \subset U$. Let $C = B^n \setminus \{\mathbf{b_0}\}$ and let

$$E_k = \bigcup_{\mathbf{b} \in B^{nk} \setminus C^k} \tau_{\mathbf{b}}(T(R, B)).$$

For any $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_\mathbf{k}) \in B^{nk} \setminus C^k$, there exists at least one $1 \le s \le k$ such that $\mathbf{b}_s = \mathbf{b}_0$. Then

$$\tau_\mathbf{b}(T(R, B)) \subset \tau_{\mathbf{b}_1 ... \mathbf{b}_{s-1}}(\tau_{\mathbf{b}_0}(T(R, B))) \subset \tau_{\mathbf{b}_1 ... \mathbf{b}_{s-1}}(U).$$

As U satisfies the open set condition for the IFS $\{\tau_b : b \in B\}$, we have $\tau_{\mathbf{b}_1 ... \mathbf{b}_{s-1}}(U) \subset U$. Hence, $E_k \subset U$. Now,

$$
\begin{aligned}
1 \ge \mu(U) \ge \mu(E_k) &= \sum_{\mathbf{b} \in B^{nk}} p_\mathbf{b} \mu(\tau_\mathbf{b}^{-1}(E_k)) \\
&\ge \sum_{\mathbf{b} \in B^{nk} \setminus C^k} p_\mathbf{b} \mu(\tau_\mathbf{b}^{-1}(E_k)) \\
&\ge \sum_{\mathbf{b} \in B^{nk} \setminus C^k} p_\mathbf{b} \mu(\tau_\mathbf{b}^{-1}(\tau_\mathbf{b}(T(R, B)))) \\
&= \sum_{\mathbf{b} \in B^{nk} \setminus C^k} p_\mathbf{b} \\
&= \sum_{\mathbf{b} \in B^{nk}} p_\mathbf{b} - \sum_{\mathbf{b} \in C^k} p_\mathbf{b} \\
&= 1 - \Big(\sum_{\mathbf{b} \in C} p_\mathbf{b}\Big)^k = 1 - (1 - p_{\mathbf{b}_0})^k.
\end{aligned}
$$

As $1 - p_{\mathbf{b}_0} > 0$, $(1 - p_{\mathbf{b}_0})^k$ tends to 0 as k tends to infinity. This shows that $\mu(U) = 1$. As $T(R, B) \subset \overline{U}$ (because $\cup_b \tau_b(\overline{U}) \subset \overline{U}$), we must have $\mu(\overline{U}) = 1$ and $\mu(\partial U) = 0$.

For the no overlap condition, we note that $T(R, B) \subset \overline{U}$. Then $T(R, B)_b \subset (\overline{U})_b = \overline{U_b}$. Hence,

$$\tau_b(T(R, B)) \cap \tau_{b'}(T(R, B)) \subseteq \overline{U_b} \cap \overline{U_{b'}} = (U_b \cap \partial(U_b)) \cup (U_{b'} \cap \partial(U_{b'})).$$

But U is an open set satisfying the OSC, so $\tau_b(T(R, B)) \cap \tau_{b'}(T(R, B)) \subseteq \partial(U_b) \cap \partial(U_{b'})$. The no overlap condition will follow if we can show that $\mu(\partial U_b) = 0$ for all $b \in B$.

Suppose on the contrary that $\mu(\partial U_b) > 0$, we apply (2.1) and obtain

$$0 < \mu(\partial U_b) = \sum_{b' \in B} p_b \mu(\tau_{b'}^{-1}(\partial U_b))$$

This implies that for some b', $\mu(\tau_{b'}^{-1}(\partial U_b)) > 0$. But $\tau_{b'}^{-1}(\partial U_b) = \partial U + Rb - b'$ and μ is supported essentially on U, so we have

$$\mu((\partial U + Rb - b') \cap U) > 0.$$

As U is open, $U \cap (U + Rb - b') \ne \emptyset$. This implies that $\tau_b \tau_0(U) \cap \tau_0 \tau_{b'}(U) \ne \emptyset$ and this contradicts to the open set condition for U (by a translation we can always assume $0 \in B$). Hence, $\mu(\partial U_b) = 0$ and this completes the proof. $\qquad \square$

As we have mentioned in the introduction, (T-SOSC) is the condition we aim to study. We will see that (T-SOSC) implies the SOSC and hence OSC. However, we don't know if they are equivalent.

Proposition 2.2. *Let R be a $d \times d$ integer expansive matrix and B be a simple digit set for R. Suppose that $\overline{B} \supset B$ is a complete representative class $(\mathrm{mod}\, R\mathbb{Z}^d)$. Then the open set condition for the IFS $\{\tau_b\}_{b \in B}$ is satisfied with open set $T(R, \overline{B})^\circ$.*

Let $\mu = \mu(R, B)$ be the invariant measure of the iterated function system. Suppose that (T-SOSC) is satisfied for $\{\tau_b\}_{b \in B}$ with $T = T(R, \overline{B})$. Then the SOSC is satisfied with open set T°; also $\mu(T^\circ) = 1$, $\mu(\partial T) = 0$, the IFS satisfies the no overlap condition and if T tiles with the lattice $\mathbb{Z}^d$, then

$$\mu((T(R, B) + n) \cap (T(R, B) + n')) = 0 \ \forall \ n, n' \in \mathbb{Z}^d.$$

Proof. The statement that the open set condition is satisfied for the IFS $\{\tau_b\}_{b \in B}$ with open set $T^\circ(R, \overline{B})$ is probably known, but we present the proof for completeness. Let $T = T(R, \overline{B})$ and note that $T = \bigcup_{b \in \overline{B}} \tau_b(T)$. By taking the interior, we have $T^\circ \supset \bigcup_{b \in \overline{B}} \tau_b(T^\circ) \supset \bigcup_{b \in B} \tau_b(T^\circ)$. Also T° is non-empty, by [LW2]. To see $\tau_b(T^\circ) \cap \tau_{b'}(T^\circ) \neq \emptyset$, we take Lebesgue measure on the invariance identity and obtain

$$\mathrm{Leb}(T) = \mathrm{Leb}\left(\bigcup_{b \in \overline{B}} \tau_b(T)\right) \leq \sum_{b \in \overline{B}} \mathrm{Leb}(\tau_b(T)) = \frac{\#\overline{B}}{|\det(R)|}\mathrm{Leb}(T) = \mathrm{Leb}(T).$$

Here $\mathrm{Leb}(T)$ denotes the Lebesgue measure of T and $\#\overline{B} = |\det(R)|$ because $\overline{B}$ is a set of complete representatives (mod $R\mathbb{Z}^d$). $\mathrm{Leb}(T)$ is non-zero, by [LW2]. Hence,

$$\mathrm{Leb}\left(\bigcup_{b \in \overline{B}} \tau_b(T)\right) = \sum_{b \in \overline{B}} \mathrm{Leb}(\tau_b(T))$$

and $\mathrm{Leb}(\tau_b(T) \cap \tau_{b'}(T)) = 0$. This implies that $\tau_b(T^\circ) \cap \tau_{b'}(T^\circ) = \emptyset$ since $\tau_b(T^\circ) \cap \tau_{b'}(T^\circ)$ is an open set.

Suppose that (T-SOSC) is satisfied for $\{\tau_b\}_{b \in B}$ with $T = T(R, \overline{B})$. By the previous property and Theorem 1.5 we get that the SOSC condition is satisfied with open set T°. The conclusion that $\mu(T^\circ) = 1$, $\mu(\partial T) = 0$ and that the IFS satisfies the no overlap condition are proved in Theorem 2.1 by taking $U = T^\circ$. To prove the last statement, we note that for all $n \in \mathbb{Z}^d$, $\mu(\partial T + n) = \mu((\partial T + n) \cap T^\circ) = 0$ (as T is a $\mathbb{Z}^d$-tile and $\mu(\partial T) = 0$). Moreover, $T(R, B) \subset T$ and T tiles by $\mathbb{Z}^d$ implies that for any $n \neq n'$ in $\mathbb{Z}^d$,

$$(T(R, B) + n) \cap (T(R, B) + n') \subseteq (T + n) \cap (T + n') = (\partial T + n) \cap (\partial T + n').$$

Hence, $\mu((T(R, B) + n) \cap (T(R, B) + n')) \leq \mu((\partial T + n) \cap (\partial T + n')) = 0.$ $\qquad\square$

Proof of Theorem 1.7. Proposition 2.2 shows that the OSC is satisfied, Theorem 1.5 shows that the SOSC is satsisfied and then Theorem 2.1 shows that the no overlap condition holds.

If (i) is satisfied, Proposition 2.2 shows that (ii) holds.

If (ii) holds, consider the set

$$\mathcal{N} := \{x \in T(R,B) : \text{ There exists } y \in T(R,B), y \neq x \text{ such that } e^{2\pi i \langle n, x-y \rangle} = 1 \text{ for all } n \in \mathbb{Z}^d\}$$

$$= \{x \in T(R,B) : \text{ There exists } y \in T(R,B) \text{ such that } -\alpha := x - y \in \mathbb{Z}^d\}$$

$$= \bigcup_{\alpha \in \mathbb{Z}^d} \{x \in T(R,B) : x + \alpha \in T(R,B)\} = \bigcup_{\alpha \in \mathbb{Z}^d} (T(R,B) \cap (T(R,B) - \alpha)).$$

By hypothesis, $\mathcal{N}$ has measure zero.

Now take $\mathcal{K}$ to be an arbitrary compact subset of $T(R,B) \backslash \mathcal{N}$. The collection of exponential functions $E(\mathbb{Z}^d) := \{e^{2\pi i \langle n, x \rangle} : n \in \mathbb{Z}^d\}$ separates points in $\mathcal{K}$, therefore, by Stone-Weierstrass theorem, we get that $E(\mathbb{Z}^d)$ spans $L^2(\mathcal{K}, \mu)$, and since $\mathcal{K}$ was arbitrary close to $T(R,B)$ in measure, we get that these exponentials span $L^2(T(R,B), \mu)$. Hence, for $\xi \in \mathbb{R}^d$ we cannot have $\widehat{\mu}(\xi + n) = 0$ for all $n \in \mathbb{Z}^d$, because that would imply that $e^{2\pi i \langle \xi, x \rangle}$ is orthogonal to all $e^{2\pi i \langle n, x \rangle}$ for all $n \in \mathbb{Z}^d$, which contradicts the completeness. This shows $\mathcal{Z} = \emptyset$.

□

3. Almost-Parseval-frame conditions and Hadamard triples.

In this section, we study the almost-Parseval-frame condition in Definition 1.10. First of all, we note that there is no loss of generality to assume $0 \in J_n$, because we can replace w_b by $w_b e^{2\pi i \langle R^{-n}b, \lambda_0 \rangle}$, and (1.5) is satisfied with J_n replaced by $J_n - \lambda_0$.

Proposition 3.1. *Suppose that the pair (R, B) satisfies the almost Parseval frame condition and $J_n \subset \mathbb{Z}^d$ is the set satisfying (1.5), with $\epsilon < 1$. We have the following:*

(i) The elements in J_n have distinct residues modulo $(R^T)^n(\mathbb{Z}^d)$.

(ii) Let $\widehat{J_n} \equiv J_n \pmod{(R^T)^n \mathbb{Z}^d}$, then $\widehat{J_n}$ also satisfies (1.5).

Proof. (i) Suppose on the contrary that we can find $\lambda', \lambda'' \in J_n$ such that λ' and λ'' are in the same equivalence class modulo $(R^T)^n \mathbb{Z}^d$. Let $w_b = e^{2\pi i \langle R^{-n}b, \lambda'' \rangle}$, for all $b \in B$, and plug it in (1.5). From the upper bound, we have

$$2N^n + \sum_{\lambda \in J_n \backslash \{\lambda', \lambda''\}} \left| \frac{1}{\sqrt{N^n}} \sum_{b \in B_n} w_b e^{2\pi i \langle R^{-n}b, \lambda \rangle} \right|^2 \leq (1+\epsilon)N^n.$$

This implies that

$$\sum_{\lambda \in J_n \setminus \{\lambda', \lambda''\}} \left| \frac{1}{\sqrt{N^n}} \sum_{b \in B_n} w_b e^{-2\pi i \langle R^{-n} b, \lambda \rangle} \right|^2 \leq (\epsilon - 1) N^n < 0$$

which is a contradiction. (ii) follows immediately from $\langle R^{-n} b, \lambda + (R^T)^n k \rangle = \langle R^{-1} b, \lambda \rangle$ for all $b \in B_n$, $\lambda \in J_n$ and $k \in \mathbb{Z}^d$. $\qquad\square$

Assuming that the almost-Parseval-frame condition is satisfied, we consider sequences ϵ_k such that $\sum_k \epsilon_k < \infty$ and let n_k and J_{n_k} be the associated quantities satisfying

$$(1 - \epsilon_k) \sum_{b \in B_{n_k}} |w_b|^2 \leq \sum_{\lambda \in J_{n_k}} \left| \sum_{b \in B_{n_k}} \frac{1}{\sqrt{N^{n_k}}} w_b e^{-2\pi i \langle R^{-n_k} b, \lambda \rangle} \right|^2 \leq (1 + \epsilon_k) \sum_{d \in \mathcal{D}_{n_k}} |w_d|^2.$$

Letting $m_k = n_1 + n_2 + ... + n_k$, we consider

$$(3.1) \qquad \Lambda_k = J_{n_1} + (R^T)^{m_1} J_{n_2} + (R^T)^{m_2} J_{n_3} + ... + (R^T)^{m_{k-1}} J_{n_k},$$

$$(3.2) \qquad \Lambda = \bigcup_{k=1}^{\infty} \Lambda_k.$$

We will see that Λ will be our candidate for the spectra in the rest of this section and the next section. Note that the digit sets $B_{m_1} \subset B_{m_2} \subset ...$ satisfy

$$B_{m_{k+1}} = R^{n_{k+1}} B_{m_k} + B_{n_{k+1}}, B_{m_1} = B_{n_1}.$$

Proposition 3.2. *With the notations above, we have*

$$c_k \|\mathbf{w}\|^2 \leq \sum_{\lambda \in \Lambda_k} \left| \frac{1}{\sqrt{N^{m_k}}} \sum_{b \in B_{m_k}} w_b e^{-2\pi i \langle R^{-m_k} b, \lambda \rangle} \right|^2 \leq C_k \|\mathbf{w}\|^2$$

where $c_k = \prod_{j=1}^{k} (1 - \epsilon_j)$ and $C_k = \prod_{j=1}^{k} (1 + \epsilon_j)$.

Proof. We prove this by induction on k. The inequality for $k = 1$ is the almost-Parseval-frame condition with B_{n_1} and J_{n_1}. Assuming the inequality is proved on k, we now establish it for $k + 1$. We consider the upper bound inequality. If $d \in B_{m_{k+1}}$ and $\lambda \in \Lambda_{k+1}$, we can write $b = R^{n_{k+1}} b_1 + b_2$ and $\lambda = \lambda_1 + (R^T)^{m_k} \lambda_2$ where $\lambda_1 \in \Lambda_k$, $\lambda_2 \in J_{n_{k+1}}$, $b_1 \in B_{m_k}$ and

$b_2 \in B_{n_{k+1}}$.

$$\sum_{\lambda \in \Lambda_{k+1}} \left| \frac{1}{\sqrt{N^{m_{k+1}}}} \sum_{d \in B_{m_{k+1}}} w_d e^{-2\pi i \langle R^{-m_{k+1}} b, \lambda \rangle} \right|^2$$

$$= \sum_{\lambda_1 \in \Lambda_k} \sum_{\lambda_2 \in J_{n_{k+1}}} \left| \frac{1}{\sqrt{N^{m_{k+1}}}} \sum_{d_2 \in B_{n_{k+1}}} \sum_{d_1 \in B_{m_k}} w_{b_1 b_2} e^{-2\pi i \langle R^{-m_{k+1}} (R^{n_{k+1}} b_1 + b_2), \lambda_1 + (R^T)^{m_k} \lambda_2 \rangle} \right|^2$$

$$= \sum_{\lambda_1 \in \Lambda_k} \sum_{\lambda_2 \in J_{n_{k+1}}} \left| \frac{1}{\sqrt{N^{n_{k+1}}}} \sum_{b_2 \in B_{n_{k+1}}} e^{-2\pi i \langle R^{-n_{k+1}} b_2, \lambda_2 \rangle} \sum_{b_1 \in B_{m_k}} \frac{1}{\sqrt{N^{m_k}}} w_{b_1 b_2} e^{-2\pi i \langle R^{-m_k} b_1 + R^{-m_{k+1}} b_2, \lambda_1 \rangle} \right|^2$$

$$\leq (1 + \epsilon_{k+1}) \sum_{\lambda_1 \in \Lambda_k} \sum_{b_2 \in B_{n_{k+1}}} \left| \frac{1}{\sqrt{N^{m_k}}} \sum_{b_1 \in B_{m_k}} w_{b_1 b_2} e^{-2\pi i \langle R^{-m_k} b_1 + R^{-m_{k+1}} b_2, \lambda_1 \rangle} \right|^2$$

$$= (1 + \epsilon_{k+1}) \sum_{b_2 \in B_{n_{k+1}}} \sum_{\lambda_1 \in \Lambda_k} \left| \frac{1}{\sqrt{N^{m_k}}} \sum_{b_1 \in B_{m_k}} w_{b_1 b_2} e^{-2\pi i \langle R^{-m_k} b_1, \lambda_1 \rangle} \right|^2 \quad \left(\text{as } \left| e^{-2\pi i \langle R^{-m_{k+1}} b_2, \lambda_1 \rangle} \right| = 1 \right)$$

$$\leq (1 + \epsilon_{k+1}) C_k \sum_{b_2 \in B_{n_{k+1}}} \sum_{b_1 \in B_{m_k}} |w_{b_1 b_2}|^2 = C_{k+1} \|\mathbf{w}\|^2.$$

The proof for the lower bound is similar.   □

Now, we turn to study Hadamard triples (R, B, L) as defined in (1.1) in the introduction. We first remark that the elements of B must be in distinct residue classed modulo $R(\mathbb{Z}^d)$, because H must have mutually orthogonal rows. This implies that

$$(3.3) \qquad\qquad \sum_{\ell \in L} e^{2\pi i \langle R^{-1}(b-b'), \ell \rangle} = 0, \text{ if } b \neq b'.$$

If $b = b' + Rk$ for some $k \in \mathbb{Z}^d$, the sum above is equal to $\#L \neq 0$. Similarly, the elements L must be in distinct residue class modulo $R^T \mathbb{Z}^d$. As H is a unitary matrix, it is clear that we have $\|H\mathbf{w}\| = \|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{C}^N$. i.e.

$$\sum_{\ell \in L} \left| \sum_{b \in B} w_b \frac{1}{\sqrt{N}} e^{-2\pi i \langle R^{-1} b, \ell \rangle} \right|^2 = \sum_{b \in B} |w_b|^2.$$

From this, we will conclude in Corollary 3.3 that (R, B) satisfies the almost-Parseval-frame condition (with $\epsilon = 0$!). We also need to consider towers of Hadamard triples. Using the

definition of B_n in (1.4) and defining similarly L_n^T by

$$L_n^T = L + R^T L + (R^T)^2 L + ... + (R^T)^{n-1} L = \left\{ \sum_{j=0}^{n-1} (R^T)^j \ell_j : \ell_j \in L \right\}$$

From Proposition 3.2, we have the following corollary.

Corollary 3.3. *(i) Suppose that (R, B, L) is a Hadamard triple. Then for all $k \geq 1$, (R^k, B_k, L_k^T) are Hadamard triples.*

(ii) Suppose that $(R^{n_i}, B_{n_i}, J_{n_i})$, $i = 1, 2, ..$, are Hadamard triples, then for all $k \geq 1$, $(R^{m_k}, B_{m_k}, \Lambda_k)$ are Hadamard triples where Λ_k are defined in (3.1).

Proof. Suppose that (R, B, L) is a Hadamard triple. Then we take $n_i = 1$ and $J_{n_i} = L$. We have $\Lambda_k = L_k^T$. Proposition 3.2 implies that

$$(3.4) \qquad \sum_{\lambda \in L_k^T} \left| \sum_{b \in B_k} \frac{1}{\sqrt{N^k}} w_b e^{-2\pi i \langle R^{-k} b, \lambda \rangle} \right|^2 = \|\mathbf{w}\|^2, \ \forall \mathbf{w} \in \mathbb{C}^{N^k}.$$

Similarly, if $(R^{n_i}, B_{n_i}, J_{n_i})$, $i = 1, 2, ..$, are Hadamard triples, we also have

$$(3.5) \qquad \sum_{\lambda \in \Lambda_k} \left| \sum_{b \in B_{m_k}} \frac{1}{\sqrt{N^{m_k}}} w_b e^{-2\pi i \langle R^{-m_k} b, \lambda \rangle} \right|^2 = \|\mathbf{w}\|^2 \ \forall \mathbf{w} \in \mathbb{C}^{N^{m_k}}.$$

From (3.4), we fix $\lambda' \in L_k^T$ and put $w_b = e^{2\pi i \langle R^{-m_k} b, \lambda' \rangle}$, for all $b \in B$. As the term in the sum that corresponds to λ' is equal to N^k, which is also $\|\mathbf{w}\|^2$, we obtain that

$$\sum_{\lambda \in L_k^T \setminus \{\lambda'\}} \left| \sum_{b \in B_k} \frac{1}{\sqrt{N^k}} e^{2\pi i \langle R^{-k} b, (\lambda' - \lambda) \rangle} \right|^2 = 0.$$

This shows that the matrix $\left[e^{2\pi i \langle R^{-k} b, \ell \rangle} \right]_{\ell \in L_k^T, b \in B_n}$ has mutually orthogonal rows and hence (R^k, B_k, L_k^T) are Hadamard triples. From a similar argument using (3.5), we obtain also that $(R^{m_k}, B_{m_k}, \Lambda_k)$ are Hadamard triples. $\qquad \square$

4. Spectral properties

We start with a proposition showing that we can always reduce our study to the case when $\mathbb{Z}[R, B] = \mathbb{Z}^d$.

Proposition 4.1. *If the lattice $\mathbb{Z}[R, B]$ is not full-rank, then the dimension can be reduced; more precisely, there exists $1 \leq r < d$ and a unimodular matrix $M \in GL(n, \mathbb{Z})$ such that $M(B) \subset \mathbb{Z}^r \times \{0\}$ and*

$$(4.1) \qquad MRM^{-1} = \begin{bmatrix} A_1 & C \\ 0 & A_2 \end{bmatrix}$$

where $A_1 \in M_r(\mathbb{Z})$, $C \in M_{r,d-r}(\mathbb{Z})$, $A_2 \in M_{d-r}(\mathbb{Z})$. In addition, $M(T(R, B)) \subset \mathbb{R}^r \times \{0\}$, if (R, B, L) is a Hadamard triple then $(MRM^{-1}, MB, (M^T)^{-1}L)$ is a Hadamard triple and the measure $\mu(R, B)$ is spectral if and only if $\mu(MRM^{-1}, MB)$ is.

If the lattice $\mathbb{Z}[R, B]$ is full rank but not $\mathbb{Z}^d$, then there exists an invertible $d \times d$ integer matrix M such that $\tilde{R} := M^{-1}RM$ is an integer matrix, $\tilde{B} := M^{-1}B$ is contained in $\mathbb{Z}^d$ and $\mathbb{Z}[\tilde{R}, \tilde{B}] = \mathbb{Z}^d$. If $\tilde{L} := M^T L$ then $(\tilde{R}, \tilde{B}, \tilde{L})$ is a Hadamard triple and $\mu(R, B)$ is spectral if and only if $\mu(\tilde{R}, \tilde{B})$ is.

Proof. If the lattice $\mathbb{Z}[R, B]$ is not full-rank, then it spans a proper rational (i.e., having a basis with rational components) subspace V of $\mathbb{R}^d$ of dimension r. Since $\mathbb{Z}[R, B]$ is invariant under R, it follows that $RV \subset V$ and since R is invertible, the dimensions must match so $RV = V$. Then there is a unimodular matrix $M \in GL(n, \mathbb{Z})$ that maps V into the first r coordinate axes, that is $MV = \mathbb{R}^r \times \{0\}$, see e.g. [Sch65, Theorem 4.1 and Corollary 4.3b]. Then also $MB \subset \mathbb{R}^r \times \{0\}$. Since

$$T(R, B) = \left\{ \sum_{n=1}^{\infty} R^{-n} b_n : b_n \in B \text{ for all } b \in B \right\},$$

we get that $T(R, B)$ is in V so $MT(R, B) \subset \mathbb{R}^r \times \{0\}$.

The subspace $\mathbb{R}^r \times \{0\}$ is invariant for MRM^{-1} and this implies that M has the form in (4.1). Since M is unimodular M^{-1} is also an integer matrix so MRM^{-1} is an integer matrix. The other statements follow by a simple computation.

If $\mathbb{Z}[R, B]$ is full rank but not $\mathbb{Z}^d$ then $\mathbb{Z}[R, B] = M\mathbb{Z}^d$ for some invertible integer matrix M. If $\{e_j\}$ are the canonical vectors in $\mathbb{R}^d$, then $RMe_j \in \mathbb{Z}[R, B]$ so $RMe_j = M\tilde{r}_j$ for some $r_j \in \mathbb{Z}^d$. So $RM = M\tilde{R}$ for an integer matrix $\tilde{R}$, i.e., $\tilde{R} = M^{-1}RM$. Since $B \subset \mathbb{Z}[R, B] = M\mathbb{Z}^d$, there exists $\tilde{B}$ in $\mathbb{Z}^d$ such that $B = M\tilde{B}$ so $\tilde{B} = M^{-1}B$. We have $M^{-1}R^k B = \tilde{R}^k \tilde{B}$ so $\mathbb{Z}[\tilde{R}, \tilde{B}] = M^{-1}\mathbb{Z}[R, B] = \mathbb{Z}^d$. The other statements follow from an easy computation. $\square$

In this section, we study the spectral properties of self-affine measures and Theorems 1.8 and 1.11 are proved. Recall that, for a given expansive integer matrix R and a set B of distinct residue modulo $R\mathbb{Z}^d$, the self-affine measures we are studying satisfy

$$\mu(E) = \sum_{b \in B} \frac{1}{N} \mu(\tau_b^{-1}(E)).$$

where $\tau_b(x) = R^{-1}(x + b)$. We only need to study measures with equal weights $\frac{1}{N}$, as there are no Fourier frames if the weights are different [DL1, Theorem 1.5]. Our goal is to show that some set Λ defined as in (3.1) and (3.2) will be a spectrum or frame spectrum for our measure.

For the self-affine measure μ, the Fourier transform can be computed by iterating the invariance identity and we have

$$\widehat{\mu}(\xi) = M_B((R^T)^{-1}\xi)\widehat{\mu}((R^T)^{-1}\xi) = \ldots = \prod_{j=1}^{n} M_B((R^T)^{-j}\xi)\widehat{\mu}((R^T)^{-n}\xi).$$

where $M_B(\xi) = \frac{1}{N}\sum_{b\in B} e^{-2\pi i\langle b,\xi\rangle}$. Note that if B_n is the set in (1.4),

$$M_{B_n}(\xi) = \frac{1}{N^n}\sum_{b\in B^n} e^{-2\pi i\langle b,\xi\rangle} = \prod_{j=0}^{n-1} M_B((R^T)^j\xi).$$

This implies that

(4.2) $$\widehat{\mu}(\xi) = M_{B_n}((R^T)^{-n}\xi)\widehat{\mu}((R^T)^{-n}\xi).$$

Lemma 4.2. *Suppose that* $(R^{n_i}, B_{n_i}, J_{n_i})$, $i = 1, 2, \ldots$, *are Hadamard triples. Then for* Λ *in (3.2), the corresponding set of exponential functions* $E(\Lambda)$ *is a mutually orthogonal set for* μ.

Proof. Note that Λ_k in (3.1) is an increasing sequence of finite sets as $0 \in J_{n_i}$ for all i. Take some distinct $\lambda, \lambda' \in \Lambda$, we choose k so that $\lambda, \lambda' \in \Lambda_k$. By Corollary 3.3, the Hadamard matrix associated to $(R^{m_k}, B_{m_k}, \Lambda_k)$ has mutually orthogonal rows and hence $M_{B_{m_k}}((R^T)^{-m_k}(\lambda - \lambda')) = 0$. By (4.2), $\widehat{\mu}(\lambda - \lambda') = 0$. $\qquad\square$

We now establish the Fourier frame inequality which implies the completeness of our set of exponentials. The idea is to consider step functions on $T(R, B)$. There is a natural one-one correspondence between B^n in Section 2 and B_n in (1.4), by identifying $(b_0, \ldots, b_{n-1})$ and $\sum_{j=0}^{n-1} R^j b_j$. With an abuse of notation, these two will be used interchangeably. Throughout the section, we assume B is a simple digit set for R, so that by Theorem 1.7, the no-overlap condition is satisfied. Let $\mathcal{S}_n$ denote the set of all step functions at level n on $T(R, B)$, i.e.,

$$\mathcal{S}_n = \left\{ \sum_{\mathbf{b}\in B^n} w_{\mathbf{b}}\mathbf{1}_{T(R,B)_{\mathbf{b}}} : w_{\mathbf{b}} \in \mathbb{C} \right\}.$$

Here $\mathbf{1}_{T(R,B)_{\mathbf{b}}}$ denotes the characteristic function of $T(R, B)_{\mathbf{b}}$. It is well known that the set

(4.3) $$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

is a dense set of $L^2(\mu)$, but we provide the proof for completeness. Moreover, by iterating the invariance equation

$$T(R,B) = \bigcup_{b \in B} \tau_b(T(R,B)),$$

it is easy to see that $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \dots$

Lemma 4.3. $\mathcal{S}$ *forms a dense set of* $L^2(\mu)$. *Suppose that* $f = \sum_{b \in B_n} w_b \mathbf{1}_{T(R,B)_{\mathbf{b}}} \in \mathcal{S}_n$ *and* $\mu = \mu(R,B)$. *Then*

$$(4.4) \qquad \int |f|^2 d\mu = \frac{1}{N^n} \sum_{b \in B_n} |w_b|^2$$

and

$$(4.5) \qquad \int f(x) e^{-2\pi i \xi x} d\mu(x) = \frac{1}{N^n} \widehat{\mu}((R^T)^{-n}\xi) \sum_{b \in B_n} w_b e^{-2\pi i \langle R^{-n} b, \xi \rangle}.$$

Proof. Take first a continuous function f on $T(R,B)$ and $\epsilon > 0$. Since $T(R,B)$ is compact, the function f is uniformly continuous. We can find m large enough such that the diameter of all sets $\tau_{\mathbf{b}}(T(R,B))$, $\mathbf{b} \in B^m$, is small enough so that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \tau_{\mathbf{b}}(T(R,B))$ and all $\mathbf{b} \in B^m$. Consider $g = \sum_{\mathbf{b} \in B^m} f(\tau_{\mathbf{b}}(0)) \mathbf{1}_{T(R,B)_{\mathbf{b}}}$. It is easy to see that $\sup_{x \in T(R,B)} |f(x) - g(x)| < \epsilon$. Hence, $\mathcal{S}$ is uniformly dense in $C(T(R,B))$. As μ is a regular Borel measure, $\mathcal{S}$ is dense in $L^2(\mu)$.

The no-overlap condition and the invariance equation for μ imply that $\mu(T(R,B)_{\mathbf{b}}) = N^{-n}$ for all $\mathbf{b} \in B^n$. This implies (4.4) immediately. To prove (4.5),

$$(4.6) \qquad \int f(x) e^{-2\pi i \xi x} d\mu(x) = \sum_{\mathbf{b} \in B^n} w_{\mathbf{b}} \int_{\tau_{\mathbf{b}}(T(R,B))} e^{-2\pi i \langle \xi, x \rangle} d\mu(x).$$

Note that

$$\int_{\tau_{\mathbf{b}}(T(R,B))} e^{-2\pi i \langle \xi, x \rangle} d\mu(x) = \frac{1}{N^n} \sum_{\mathbf{b}' \in B^n} \int \mathbf{1}_{\tau_{\mathbf{b}}(T(R,B))}(\tau_{\mathbf{b}'}(x)) e^{-2\pi i \langle \xi, \tau_{\mathbf{b}'}(x) \rangle} d\mu(x).$$

By the no overlap condition, the only non-zero term in the summation above is the one corresponding to $\mathbf{b} = \mathbf{b}'$. This yields that

$$\int_{\tau_{\mathbf{b}}(T(R,B))} e^{-2\pi i \langle \xi, x \rangle} d\mu(x) = \frac{1}{N^n} \int e^{-2\pi i \langle \xi, \tau_{\mathbf{b}}(x) \rangle} d\mu(x) = \frac{1}{N^n} e^{-2\pi i \langle \xi, R^{-n} \mathbf{b} \rangle} \widehat{\mu}((R^T)^{-n}\xi).$$

Combining with (4.6), we obtain (4.5). $\qquad \square$

For the sets Λ_k and Λ we defined in (3.1), (3.2). We consider the following quantity.

$$(4.7) \qquad \delta(\Lambda) = \inf_k \inf_{\lambda \in \Lambda_k} |\widehat{\mu}((R^T)^{-m_k}\lambda)|^2$$

Theorem 4.4. *Suppose that B is a simple digit set for R. Let $\mu = \mu(R, B)$ be the associated self-affine measure with equal weights. Assume that the almost-Parseval-frame condition is satisfied and that $\delta(\Lambda) > 0$. Then, with the notations in (3.1),(3.2) and (4.7), the set $E(\Lambda) := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is a Fourier frame for $L^2(\mu)$ with*

$$(4.8) \qquad c\delta(\Lambda)\|f\|^2 \le \sum_{\lambda \in \Lambda} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 \le C\|f\|^2$$

where $c = \prod_{j=1}^{\infty}(1 - \epsilon_j)$ and $C = \prod_{j=1}^{\infty}(1 + \epsilon_j)$.

If in addition $(R^{n_i}, B^{n_i}, J_{n_i})$ are Hadamard triples, then $E(\Lambda)$ is a spectrum of $L^2(\mu)$.

Proof. As we know that $\mathcal{S} := \bigcup_{k=1}^{\infty} \mathcal{S}_k$ forms a dense family of sets in $L^2(\mu)$ and $\mathcal{S}_k$ is an increasing sequence of collections of functions, It suffices to show that the frame inequality is true for all functions in $\mathcal{S}_{m_k}$ for the numbers m_k we defined in (3.1). By Proposition 3.2, for any $k \ge 1$

$$c_k \sum_{b \in B_{m_k}} |w_b|^2 \le \sum_{\lambda \in \Lambda_k} \left| \frac{1}{\sqrt{N^{m_k}}} \sum_{b \in B_{m_k}} w_b e^{-2\pi i \langle R^{-m_k} b, \lambda \rangle} \right|^2 \le C_k \sum_{b \in B_{m_k}} |w_b|^2$$

where $c_k = \prod_{j=1}^{k}(1 - \epsilon_j)$ and $C_k = \prod_{j=1}^{k}(1 + \epsilon_j)$. In view of Lemma 4.3,

$$\sum_{\lambda \in \Lambda_{m_k}} \left| \int f(x) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) \right|^2 = \frac{1}{N^{m_k}} \left| \frac{1}{\sqrt{N^{m_k}}} \widehat{\mu}((R^T)^{-m_k} \lambda) \sum_{b \in B_{m_k}} w_d e^{-2\pi i \langle R^{-n} b, \lambda \rangle} \right|^2.$$

As $\delta(\Lambda) \le |\widehat{\mu}((R^T)^{-m_k}\lambda)|^2 \le 1$, Lemma 4.3 implies that this term is bounded above by $C\|f\|^2$ and bounded below by $c\delta(\Lambda)\|f\|^2$,

$$c\delta(\Lambda)\|f\|^2 \le \sum_{\lambda \in \Lambda_{m_k}} \left| \int f(x) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) \right|^2 \le C\|f\|^2.$$

But since $\mathcal{S}_{m_k} \subset \mathcal{S}_{m_\ell}$ for any $\ell \ge k$, we will have

$$c\delta(\Lambda)\|f\|^2 \le \sum_{\lambda \in \Lambda_{m_\ell}} \left| \int f(x) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) \right|^2 \le C\|f\|^2, \ f \in \mathcal{S}_{m_k}.$$

This shows the frame inequality holds by letting ℓ go to infinity. $\qquad \square$

Proposition 4.5. *Let B be a simple digit set for R. Assume that the self-affine measure $\mu = \mu(R, B)$ satisfies the almost-Parseval-frame condition and that*

$$\mathcal{Z} := \{\xi \in \mathbb{R}^d : \widehat{\mu}(\xi + k) = 0, \ for \ all \ k \in \mathbb{Z}^d\} = \emptyset.$$

Then there exists sets (J_{n_i}) such that Λ_k and Λ of the form (3.1) and (3.2) such that the number in (4.7), $\delta(\Lambda) > 0$. If in addition (R, B, L) is a Hadamard triple, then the sets (J_{n_i}) can be chosen so that $(R^{n_i}, B_{n_i}, J_{n_i})$ are Hadamard triples for all i.

Assuming this proposition, Theorems 1.8 and 1.11 can be proved.

Proof of Theorem 1.8 and 1.11. To prove Theorem 1.8, suppose first that $\mathcal{Z} = \emptyset$. We take the sets (J_{n_i}) in Proposition 4.5 so that $(R^{n_i}, B_{n_i}, J_{n_i})$ are Hadamard triples and $\delta(\Lambda) > 0$. Then, by Lemma 4.2, Λ is a mutually orthogonal set and is in $\mathbb{Z}^d$. The corresponding set of exponentials is also complete because we have the the lower frame bound in (4.8) in Theorem 4.4. Conversely, if $\mathcal{Z} \neq \emptyset$, then there exists $\xi_0 \in \mathcal{Z}$ such that $\widehat{\mu}(\xi_0 + k) = 0$ for all $k \in \mathbb{Z}^d$. Denote $e_\xi(x) = e^{2\pi i \langle \xi, x \rangle}$. We have

$$\langle e_{\xi_0}, e_k \rangle = 0, \ \forall \ k \in \mathbb{Z}^d$$

This means that the exponentials $E(\Lambda)$ cannot be complete in $L^2(\mu)$ whenever Λ is a subset of $\mathbb{Z}^d$. Hence, there is no spectrum in $\mathbb{Z}^d$ for μ.

Similarly, Theorem 1.11, follows immediately from Proposition 4.5 and Theorem 4.4. $\square$

The proof of Proposition 4.5 involves the following lemma.

Lemma 4.6. *Suppose that $\mathcal{Z} = \emptyset$ and let X be any compact set on $\mathbb{R}^d$. Then there exist $\epsilon_0 > 0$, $\delta_0 > 0$ such that for all $x \in X$, there exists $k_x \in \mathbb{Z}^d$ such that for all $y \in \mathbb{R}^d$ with $\|y\| < \epsilon_0$, we have $|\widehat{\mu}(x + y + k_x)|^2 \geq \delta_0$. In addition, we can choose $k_0 = 0$ if $0 \in X$.*

Proof. As $\mathcal{Z} = \emptyset$, for all $x \in X$ there exists $k_x \in \Gamma$ such that $\widehat{\mu}(x + k_x) \neq 0$. Since $\widehat{\mu}$ is continuous, there exists an open ball $B(x, \epsilon_x)$ and $\delta_x > 0$ such that $|\widehat{\mu}(y + k_x)|^2 \geq \delta_x$ for all $y \in B(x, \epsilon_x)$. Since X is compact, there exist $x_1, \ldots, x_m \in X$ such that

$$X \subset \bigcup_{i=1}^{m} B(x_i, \frac{\epsilon_{x_i}}{2}).$$

Let $\delta := \min_i \delta_{x_i}$ and $\epsilon := \min_i \frac{\epsilon_{x_i}}{2}$. Then, for any $x \in X$, there exists i such that $x \in B(x_i, \frac{\epsilon_{x_i}}{2})$. If $\|y\| < \epsilon$, then $x + y \in B(x_i, \epsilon_{x_i})$, so $|\widehat{\mu}(x + y + k_{x_i})|^2 \geq \delta$, we can redefine k_x to be k_{x_i} to obtain the conclusion. Clearly, we can choose $k_0 = 0$ if $0 \in X$ since $\widehat{\mu}(0) = 1$. $\square$

Proof of Proposition 4.5. Let $\overline{L}$ be a complete set of representatives (mod $R^T(\mathbb{Z}^d)$) and let $X = T(R^T, \overline{L})$. If we have the Hadamard triple (R, B, L) then we pick $\overline{L}$ such that $\overline{L} \supset L$. Since the almost-Parseval-frame condition is satisfied, we can pick the sets J_{n_i} as in (3.1) and (3.2), with bounds $1 - \epsilon_i, 1 + \epsilon_i$ and $\sum_i \epsilon_i < \infty$, the elements of J_{n_i} are in distinct residue classes (mod $(R^T)^{n_i}\mathbb{Z}^d$) by Proposition 3.1(i). By Proposition 3.1(ii), we

may assume $J_{n_i} \subset \overline{L} + R^T\overline{L} + ... + (R^T)^{n_i-1}\overline{L}$, and under the Hadamard condition, we can pick $J_{n_i} = L + R^T L + ... + (R^T)^{n_i-1}L$, by Corollary 3.3. Thus,

$$(R^T)^{-(n_i+p)}J_{n_i} \subset X, \quad (p \geq 0).$$

Moreover, we can assume also that n_i is as large as we want, by using a Λ_k instead of J_{n_i}, according to Proposition 3.2, and in the case of a Hadamard triple, by Corollary 3.3.

Fix the ϵ_0 and δ_0 in Lemma 4.6. We now construct the sets Λ_k and Λ as in (3.1) and (3.2), but we replace the sets J_{n_k} by some sets $\widehat{J}_{n_k}$ to guarantee that the number $\delta(\Lambda)$ in (4.7) is positive.

We first start with $\Lambda_0 := \{0\}$ and $m_0 = n_0 = 0$. Assuming that Λ_k has been constructed, we first choose our $n_{k+1} > n_k$ so that

(4.9) $$\|(R^T)^{-(n_{k+1}+p)}\lambda\| < \epsilon_0, \ \forall \ \lambda \in \Lambda_k, p \geq 0.$$

We then define $m_{k+1} = m_k + n_{k+1}$ and

$$\Lambda_{k+1} = \Lambda_k + (R^T)^{m_k}\widehat{J_{n_{k+1}}}$$

where

$$\widehat{J_{n_{k+1}}} = \{j + (R^T)^{n_{k+1}}k(j) : j \in J_{n_k}, \ k(j) \in \mathbb{Z}^d\}$$

where $k(j)$ is chosen to be k_x from Lemma 4.6, with $x = (R^T)^{-n_{k+1}}j \in X$. As $0 \in J_{n_k}$ and $k_0 = 0$ for all k, the sets Λ_k are of the form (3.1) and form an increasing sequence. For these sets Λ_k, we claim that the associated Λ in (3.2) satisfies $\delta(\Lambda) > 0$.

To justify the claim, we note that if $\lambda \in \Lambda_k$, then

$$\lambda = \lambda' + (R^T)^{m_{k-1}}j + (R^T)^{m_k}k(j),$$

where $\lambda' \in \Lambda_{k-1}, j \in J_{n_k}$. This means that

$$(R^T)^{-m_k}\lambda = (R^T)^{-m_k}\lambda' + (R^T)^{-n_k}j + k(j).$$

By (4.9), $\|(R^T)^{-m_k}\lambda'\| < \epsilon_0$. From Lemma 4.6, since $(R^T)^{-n_k}j \in X$, we must have $|\widehat{\mu}((R^T)^{-m_k}\lambda)|^2 \geq \delta_0 > 0$. As δ_0 is independent of k, the claim is justified and hence this completes the proof of the proposition. $\quad\square$

Proof of Corollary 1.9. Take $\overline{B} = \prod_{i=1}^d\{0, 1, \ldots, N_i - 1\}$. Then $\overline{B}$ is a complete set of representatives (mod $R(\mathbb{Z}^d)$), $B \subset \overline{B}$. Also $T(R, \overline{B}) = [0,1]^d$ so it is a $\mathbb{Z}^d$ tile. We can assume that $0 \in B$ and also that for each component $i \in \{1, \ldots, d\}$ there is a $b_i \in B$ such that the i-th component of b_i is non-zero; otherwise, we can reduce the dimension by considering only those components where such a b_i exists. Then $R^{-1}b_1 + R^{-2}b_2 + \cdots + R^{-d}b_d \in T(R, B) \cap T(R, \overline{B})^\circ$, since all components are different than 0 or 1. Using Theorem 1.7 and 1.8 we obtain the result. $\quad\square$

5. Examples

In this section we give several examples to illustrate our results. First, we recover all the known results for $\mathbb{R}^1$. Now let $|R| \geq 2$ be an integer and $B \subset \mathbb{Z}$. There is no loss of generality to assume that $\gcd(B) = 1$, otherwise $\mathbb{Z}[R, B] \neq \mathbb{Z}$.

Example 5.1. Suppose that R is an integer and (R, B, L) forms a Hadamard triple on $\mathbb{R}^1$ with $\gcd(B) = 1$. Then the associated self-similar measure $\mu(R, B)$ satisfies $\mathcal{Z} = \emptyset$, with $\mathcal{Z}$ defined in (1.3), and is spectral with a spectrum in $\mathbb{Z}$.

Proof. We can assume $0 \in B$. Suppose that $\mathcal{Z} \neq \emptyset$. As $\widehat{\mu}(0) = 1$, $\mathcal{Z} \cap \mathbb{Z} = \emptyset$. Then we pick $\xi_0 \in \mathcal{Z}$ and $\xi_0 \notin \mathbb{Z}$. We claim the following fact is true:

$$(5.1) \qquad M_B(\tau_\ell(\xi_0)) \neq 0, \ \ell \in L \ \Rightarrow \ \tau_\ell(\xi_0) \in \mathcal{Z}.$$

Indeed, by considering k of the form $\ell + Re$ and $e \in \mathbb{Z}$, we have

$$0 = \widehat{\mu}(\xi_0 + k) = M_B(R^{-1}(\xi_0 + \ell + Re))\widehat{\mu}(R^{-1}(\xi_0 + \ell + Re))$$
$$= M_B(\tau_\ell(\xi_0))\widehat{\mu}(\tau_\ell(\xi_0) + e)$$

As $M_B(\tau_\ell(\xi_0)) \neq 0$, we must have $\widehat{\mu}(\tau_\ell(\xi_0) + e) = 0$ and hence $\tau_\ell(\xi_0) \in \mathcal{Z}$. With this fact in mind, we define $Y_0 = \{\xi_0\}$ and define inductively the set Y_n by

$$Y_n = \{\tau_\ell(\xi) : \ell \in L, \ \xi \in Y_{n-1}, \ M_B(\tau_\ell(\xi)) \neq 0\}.$$

By (5.1), $Y_n \subset \mathcal{Z}$ and $Y_n \cap \mathbb{Z} = \emptyset$. From the fact that (R, B, L) is a Hadamard triple, we have

$$(5.2) \qquad \sum_{\ell \in L} |M_B(\tau_\ell(\xi))|^2 \equiv 1.$$

This means that all the sets Y_n are non-empty. Also if $\xi_n \in Y_n$, then $\xi_n = \tau_{\ell_n} \circ \dots \circ \tau_{\ell_1}(\xi_0) = R^{-n}(\xi_0 + \ell_1 + \dots + R^{n-1}\ell_n)$. This means $|\xi_n| \leq |\xi_0| + D$, where $D = \mathrm{diam}(T(R, L))$. Hence, $\sup_n\{|\xi_n| : \xi_n \in Y_n\}$ is bounded. We also notice that for different $l_0 l_1 \dots l_n \neq l'_0 l'_1 \dots l'_n$ the corresponding ξ_n and ξ'_n are different, since L is a simple digit set for R. Therefore the cardinality of Y_n is increasing.

On $\mathbb{R}^1$, $\widehat{\mu}$ has only finitely many zeros in a bounded set. Therefore, there exists n_0 such that for all $n \geq n_0$, the cardinality of Y_n becomes a constant. This means that when $n \geq n_0$, each ξ_n has only one offspring $\xi_{n+1} = \tau_{l_0}(\xi_n)$, i.e , there is only one $l_0 \in L$ such that $M_B(\tau_{l_0}(\xi_n)) \neq 0$ and so $M_B(\tau_\ell(\xi_n)) = 0$ for all $\ell_n \neq 0$. From (5.2), $|M_B(\tau_{l_0}(\xi_n))| = \left|\frac{1}{N}\sum_{b \in B} e^{2\pi i b \tau_\ell(\xi_n)}\right| = 1$. This implies we have equality in a triangle inequality, and since $0 \in B$, we get that $b\tau_{l_0}(\xi_n) \in \mathbb{Z}$ for all $b \in B$. As $\gcd(B) = 1$, we can take $m_b \in \mathbb{Z}$ such $\sum_{b \in B} bm_b = 1$ and this forces $\tau_{l_0}(\xi_n) = \sum_{b \in B} m_b(b\tau_{l_0}(\xi_n)) \in \mathbb{Z}$. This is a contradiction. $\square$

We now turn to some higher dimensional examples. We consider the *d-dimensional generalized Sierpinski gasket*, which is the self-similar set in $\mathbb{R}^d$ generated by the diagonal matrix $R = \mathrm{diag}(2, ..., 2) = 2I_d \in M_d(\mathbb{Z})$ and $B = \{0, e_1, ..., e_d\}$. We say that a $d \times d$ matrix H is a *real Hadamard matrix* if H has entries chosen from ± 1 and $H^*H = dI$. It is known that if a $d \times d$ real Hadamard matrix exists, then $d = 1, 2$ or $d \equiv 0 \pmod 4$. However, the converse is still an open problem. One can refer to [MP, Chapter 9] for an account of real Hadamard matrices. By a simple multiplication by ± 1, we can always assume the first row and the first columns of H are all 1.

Example 5.2. Let d be a positive integer such that a $(d + 1) \times (d + 1)$ real Hadamard matrix exists. Then the equal-weight self-similar measure supported on the d-dimensional generalized Sierpinski gasket is spectral with a spectrum in $\mathbb{Z}^d$.

Proof. Let $H = [h_{i,j}]$ be the associated $d + 1$ real Hadamard matrix. For $j = 0, 1, ..., d$, we define vectors $\ell_0 = 0$, and $\ell_j \in \mathbb{Z}^d$, $1 \le j \le d$, by

$$\ell_{j,i} = \begin{cases} 0, & h_{i,j} = 1; \\ 1, & h_{i,j} = -1. \end{cases} \quad i = 1, ..., d.$$

In this way, $\langle R^{-1}e_i, \ell_j \rangle = (1/2)\ell_{j,i}$ and the matrix $[e^{2\pi i \langle R^{-1}e_i, \ell_j \rangle}] = H$. This shows (R, B, L) is a Hadamard triple. As $B \subset \overline{B} = \{0, 1\}^d$, our conclusion follows from Corollary 1.9. $\square$

The following example was considered in [DJ2], where the authors found some spectra but none contained in $\mathbb{Z}^2$. We show here that it is a spectral measure with a spectrum in $\mathbb{Z}^2$.

Example 5.3. Let $R = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$,

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \text{ and } L = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

Then (R, B, L) forms a Hadamard triple and the IFS satisfies (T-SOSC) with T a $\mathbb{Z}^2$-tile. Consequently, $\mu(R, B)$ is a spectral measure with a spectrum in $\mathbb{Z}^2$.

Proof. The fact that (R, B, L) is a Hadamard triple follows from a direct check. Also, it is easy to see that $\mathbb{Z}[R, B] = \mathbb{Z}^2$. By Theorem 1.7 and 1.8, we only need to see (T-SOSC) is satisfied with T a $\mathbb{Z}^2$-tile. Consider

$$\overline{B} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : i, j \in \{0, 1, 2, 3\} \right\}.$$

Clearly, $B \subset \overline{B}$. This kind of self-affine tiles $T(R, \overline{B})$ were studied in [DeL]. It is proved there that $T(R, \overline{B})$ is a $\mathbb{Z}^2$-tile ([DeL, Proposition 2.2]) and it is homeomorphic to the disk, which

we call *disk-like* ([DeL, Theorem 1.1]). Indeed, by expanding $T(R, \overline{B}) = \{\sum_{n=1}^{\infty} R^{-n} b_n : b_n \in \overline{B}\}$, it is easy to obtain that

$$T(R, \overline{B}) = \left\{ \begin{bmatrix} \sum_{n=1}^{\infty} 4^{-n} i_n \\ g(\{i_n\}) + \sum_{n=1}^{\infty} 4^{-n} j_n \end{bmatrix} : i_n, j_n \in \{0, 1, 2, 3\} \right\} = \bigcup_{x \in [0,1]} \{x\} \times ([0, 1] + g(x))$$

where $g : [0, 1] \to \mathbb{R}$ is a measurable function obtaining from the sub-diagonal entries of R^{-n}, $n \geq 1$. On the other hand,

$$T(R, B) = \left\{ \begin{bmatrix} \sum_{n=1}^{\infty} 4^{-n} i_n \\ g(\{i_n\}) + \sum_{n=1}^{\infty} 4^{-n} j_n \end{bmatrix} : i_n \in \{0, 1\}, j_n \in \{0, 3\} \right\} = \bigcup_{x \in K_1} \{x\} \times (K_3 + g(x)),$$

where K_1 is the one-fourth Cantor set with digit $\{0, 1\}$ and K_3 is the one-fourth Cantor set with digit $\{0, 3\}$. As the tile is homeomorphic to the disk, by comparing the above two expressions, we obtain that $T(R, \overline{B})$ is the tile for the (T-SOSC) of $T(R, B)$. $\square$

The following example shows however that $\mathcal{Z} = \emptyset$ does not necessarily hold. Moreover, the measure does not admit any spectrum in $\mathbb{Z}^2$.

Example 5.4. Let $R = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$,

$$B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \text{ and } L = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Then (R, B, L) forms a Hadamard triple and $\mathbb{Z}[R, B] = \mathbb{Z}^2$. However, the set defined in (1.3) $\mathcal{Z} \neq \emptyset$ for the measure $\mu = \mu(R, B)$. Hence, (T-SOSC) is not satisfied with T a $\mathbb{Z}^2$-tile. Nonetheless, $\mu(R, B)$ is still a spectral measure, but there is no spectrum in $\mathbb{Z}^2$.

Proof. It is a direct check to see (R, B, L) forms a Hadamard triple and $\mathbb{Z}[R, B] = \mathbb{Z}^2$. As $M_B(\xi_1, \xi_2) = \frac{1}{4}(1 + e^{2\pi i \xi_1})(1 + e^{2\pi i 3 \xi_2})$. It follows that the zero set of M_B, denoted by $Z(M_B)$, is equal to

$$Z(M_B) = \left\{ \begin{bmatrix} \frac{1}{2} + n \\ y \end{bmatrix} : n \in \mathbb{Z}, y \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} x \\ \frac{1}{6} + \frac{1}{3}n \end{bmatrix} : x \in \mathbb{R}, n \in \mathbb{Z} \right\}.$$

Let $(R^T)^j = \begin{bmatrix} 4^j & a_j \\ 0 & 2^j \end{bmatrix}$, for some $a_j \in \mathbb{Z}$. As $\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} M_B((R^T)^{-j}(\xi))$, the zero set of $\widehat{\mu}$, denoted by $Z(\widehat{\mu})$, is equal to

$$Z(\widehat{\mu}) = \bigcup_{j=1}^{\infty} (R^T)^j Z(M_B)$$

$$= \bigcup_{j=1}^{\infty} \left\{ \begin{bmatrix} 4^j(\frac{1}{2} + n) + a_j y \\ 2^j y \end{bmatrix} : n \in \mathbb{Z}, y \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 4^j x + a_j(\frac{1}{6} + \frac{1}{3}n) \\ 2^j \left(\frac{1}{6} + \frac{1}{3}n \right) \end{bmatrix} : x \in \mathbb{R}, n \in \mathbb{Z} \right\}.$$

We claim that the points in $\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} + \mathbb{Z}^2$ are in $Z(\widehat{\mu})$ which shows $\mathcal{Z} \neq \emptyset$. Indeed, for any

$\begin{bmatrix} m \\ \frac{1}{3} + n \end{bmatrix}$, $m, n \in \mathbb{Z}$, we can write it as $\begin{bmatrix} m \\ \frac{1+3n}{3} \end{bmatrix}$. We now rewrite the second union in $Z(\widehat{\mu})$

as $\mathbb{R} \times \{\frac{2^{j-1}(1+2n)}{3}\}$. As any integer can be written as $2^j p$, for some $j \geq 0$ and odd number

p, this means that $\begin{bmatrix} m \\ \frac{1+3n}{3} \end{bmatrix} \in Z(\widehat{\mu})$, justifying the claim. As $\mathcal{Z} \neq \emptyset$, this shows that there is

no spectrum in $\mathbb{Z}^2$ for this measure. By Theorem 1.7, (T-SOSC) cannot be satisfied with T a $\mathbb{Z}^2$-tile. However, this Hadamard triple satisfies the reducibility condition in [DJ2]. It is a spectral measure.

$\square$

Remark 5.5. In Example 5.4, the IFS does not satisfy the (T-SOSC) with T a $\mathbb{Z}^2$-tile. However, it does satisfy the (T-SOSC) if we choose

$$\overline{B} = \left\{ \begin{bmatrix} i \\ 3j \end{bmatrix} : i \in \{0, 1, 2, 3\}, \ j \in \{0, 1\} \right\}.$$

As $\overline{B}$ is a quasi-product form digit set for R ([LW2, Section 5]), the tile $T(R, \overline{B})$ is a self-affine tile with a lattice tiling set $\mathbb{Z} \times 3\mathbb{Z}$. To see its interior contains $T(R, B)$, a simple check allows us to see that

$$T(R, \overline{B}) = \bigcup_{x \in [0,1]} \{x\} \times (g(x) + [0, 3]), \ T(R, B) = \bigcup_{x \in K_1} \{x\} \times (g(x) + [0, 3])$$

where $g : [0, 1] \to \mathbb{R}$ is a measurable function obtaining from the off-diagonal entries. Moreover,

$$T(R, \overline{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} T(\widetilde{R}, D)$$

where $\widetilde{R} = \begin{bmatrix} 4 & 0 \\ 1/3 & 2 \end{bmatrix}$ and $D = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : i \in \{0, 1, 2, 3\}, \ j \in \{0, 1\} \right\}$. As $T(\widetilde{R}, D)$ is disk-like (by Theorem 1.1 in [DeL]), $T(R, \overline{B})$ is also disk-like. This shows the interior contains $T(R, B)$.

6. SOME DISCUSSIONS AND OPEN QUESTIONS

In this section we discuss some open questions that we believe are interesting for further research and would lead to complete solutions for the problems we formulated in Section 1. Let us summarize Theorem 1.7 and 1.8 in the following implications:

(T-SOSC) and T is a $\mathbb{Z}^d$ tile $\implies \mathcal{Z} = \emptyset \iff \mu$ is spectral with spectrum in $\mathbb{Z}^d$.

From the purely fractal geometric point of view, regardless of the Fourier analytic part, (T-SOSC) is an interesting geometric condition and the answer to following question is not known:

Question 3: Does every affine IFS with a simple digit set satisfy the (T-SOSC)?

Concerning the Conjecture 1.3, we need to answer the following question:

Question 4: Suppose that (R, B, L) is a Hadamard triple and $\mathbb{Z}[R, B] = \mathbb{Z}^d$. Suppose furthermore that $\mathcal{Z} \neq \emptyset$. Is the measure $\mu(R, B)$ spectral?

One of the cases when there is a positive answer to Question 4 is when the set B is a complete set of representatives (mod $R(\mathbb{Z}^d)$) and in this case $\mu(R, B)$ is the renormalized Lebesgue measure supported on the self-affine tile. Indeed, examining the proof of [LW2, Section 6], one can reformulate Lagarias and Wang's result as: if $\mathcal{Z} \neq \emptyset$, then B is of quasi-product form. However, their proof relies strongly on the fact that B is a group, and in general, for us, the set B in Question 4 is not so. This obstructs us from obtaining any nice structure on B.

Concerning the construction of Fourier frames on fractal measures, Theorem 1.11 points toward the following question.

Question 5: Suppose that (R, B) (with B a simple digit set for R) satisfies $\mathcal{Z} = \emptyset$. When does (R, B) satisfy the almost-Parseval-frame condition?

It is not clear at the moment whether almost-Parseval-frame condition can be satisfied for other sets than those that give Hadamard triples, but the solution of the Kadison-Singer problem gives us some evidence. Let A be an $K \times L$ matrix and $J \subset \{1, ..., K\}$, we denote by $A(J)$ the sub-matrix of A whose rows belong to the index J. Nitzan et al. derived the following lemma from a version of the Kadison-Singer problem.

Lemma 6.1. [NOU, Lemma 3] *There exists universal constant $c_0, C_0 > 0$ such that whenever A is a $K \times L$ matrix, which is a sub-matrix of some $K \times K$ orthonormal matrix, such that all of its rows have equal ℓ^2-norm, one can find a subset $J \subset \{1, ..., K\}$ such that*

$$c_0 \frac{L}{K} \|\mathbf{w}\|^2 \leq \|A(J)\mathbf{w}\|^2 \leq C_0 \frac{L}{K} \|\mathbf{w}\|^2, \ \forall \mathbf{w} \in \mathbb{C}^n.$$

This lemma leads naturally to the proof of Proposition 1.12.

Proof of Proposition 1.12. Let

$$\mathcal{F}_n = \frac{1}{|\det R|^{n/2}} \left[e^{2\pi i \langle R^{-n} b, \ell \rangle} \right]_{\ell \in \overline{L}_n, b \in \overline{B}_n}$$

where $\overline{B}_n$ is a complete coset representative (mod $R(\mathbb{Z}^d)$) containing B_n and $\overline{L}_n$ is a complete coset representative (mod $R^T(\mathbb{Z}^d)$). It is well known that $\mathcal{F}_n$ is an orthonormal matrix. Let $K = |\det R|^n$ and

$$A_n = \frac{1}{|\det R|^{n/2}} \left[e^{2\pi i \langle R^{-n} b, \ell \rangle} \right]_{\ell \in \overline{L}_n, b \in B_n}.$$

Then A_n is a sub-matrix of $\mathcal{F}_n$ whose column are exactly B_n so that the size L is $L = N^n$. By Lemma 6.1, we can find a universal constant c_0, C_0, independent of n, such that for some $J_n \subset \overline{L}_n$, we have

$$c_0 \frac{N^n}{|\det R|^n} \|\mathbf{w}\|^2 \leq \|A(J_n)\mathbf{w}\|^2 \leq C_0 \frac{N^n}{|\det R|^n} \|\mathbf{w}\|^2, \ \forall \mathbf{w} \in \mathbb{C}^{N^n}.$$

As $\frac{|\det R|^{n/2}}{N^{n/2}} A(J_n) = \frac{1}{|\det R|^{n/2}} \left[e^{2\pi i \langle R^{-n} b, \ell \rangle} \right]_{\ell \in J_n, b \in B_n} := F_n$, this shows

$$c_0 \|\mathbf{w}\|^2 \leq \|F_n \mathbf{w}\|^2 \leq C_0 \|\mathbf{w}\|^2, \ \forall \mathbf{w} \in \mathbb{C}^{N^n}.$$

This is equivalent to the inequality we stated. $\qquad\square$

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